Supersymmetric quantum mechanical models with continuous spectrum and the Witten index

1985 J. Phys. A: Math. Gen. 18 3151

(http://iopscience.iop.org/0305-4470/18/16/017)

J. Phys. A: Math. Gen. 18 (1985) 3151-3156. Printed in Great Britain

Supersymmetric quantum mechanical models with continuous spectrum and the Witten index

H W Braden and A J Macfarlane

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

Received 22 March 1985

Abstract. We discuss supersymmetric regularisations of soluble supersymmetric quantum mechanical models with continuous parts to the energy spectra. Results include a demonstration that the Witten index changes discontinuously from zero to one upon removal of the regularisation.

1. Introduction

The study of supersymmetric quantum mechanics was initiated and developed extensively by Witten (1981, 1982) as a vital element in the understanding of the properties of supersymmetric field theories in two and four dimensions. Since then the study has developed extensively along a variety of lines. One line (Salomonson and van Holten 1982, Kihlberg *et al* 1984, Cecotti and Girardello 1982, Girardello *et al* 1983, Niemi and Wijewardhana 1984) concerns the study of the Witten index as an indicator of supersymmetry breaking in field theory. Another line explores the relationship to stochastic quantisation (Cooper and Freedman 1983) and to the closely related matter of the Nicolai mapping (Cecotti and Girardello 1982, Damgaard and Tsokos 1984). There is also the study, motivated from supergravity theory, of the supersymmetric quantum mechanics of σ models of both compact (Witten 1981, Davis *et al* 1983b, 1984a, b) and non-compact type (Davis *et al* 1983b, 1984a). One aspect of supersymmetric quantum mechanical models that is of interest is that they are often soluble, which makes them useful as a testing ground for ideas relevant to higher dimensional theories. Further, they may be of direct relevance to the analysis of problems in solid state physics, including spin systems (Khare and Maharana 1984).

In this paper, we introduce a new family of soluble supersymmetric quantum mechanical models. They are periodic in space, being defined on the circle C_K of circumference $2K$. They are analogues of completely integrable theories on the real line with reflectionless potentials, theories which actually arise from our models in the infinite K limit. Indeed the interrelationships between supersymmetric quantum mechanics and integrable systems is developed in our work below.

A further reason for studying the models can be described as follows. Our models illustrate the effects on the structure of a theory of a regularisation procedure that is widely employed in field theory, in which a spatial cutoff is used to make the spectra of operators of the theory discrete. We highlight the effects of the regularisation procedure by calculating the Witten index of our models. We would stress that in our

0305-4470/85/163151+06\$02.25

work supersymmetry is maintained throughout all stages of the regularisation procedure. In our regularised theories on C_K, we find that there exist two discrete ground states of energy $E=0$, which are each invariant under the supersymmetry generators, but one of them is bosonic and one fermionic so that $\Delta=0$. Secondly, and as explained below, $\Delta=0$ for all finite K, but $\Delta=1$ when K passes to its limiting value of ∞ upon removal of the regularisation of the original theory with the continuous spectrum.

This discontinuous change can be understood from various perspectives. In what is perhaps the simplest, the asymptotic behaviour of the potential changes discontinuously from periodic to non-periodic, and so the Witten index can also change discontinuously. In the conclusion we mention how this may be viewed in terms of the index theorem.

Since this work was completed several papers have appeared describing the Witten index for theories with continuous spectra. Imbimbo and Mukhi (1984a, b) utilise the aforementioned connection with the index theorem. They calculate $\Delta_B = \mathrm{Str}\exp(-\beta H)$, where Str is an appropriately defined graded trace. While the removal of the β cutoff in Euclidean time yields a continuous function of β, their analysis depends crucially on the assumption that the Hamiltonian is a Fredholm operator which restricts the class of models severely. Another approach to calculating the Witten index is via the study of the Nicolai map (1980) which can be exhibited for these quantum mechanical models (and indeed the $d=2$, $N=2$ supersymmetric field theory (Cecotti and Girardello 1982)). Cecotti and Girardello (1984) utilise this approach. It should be noted that because we have an explicit superpotential for the models on the circle C_K, this may be viewed as giving the Nicolai map in the usual way, and the discontinuity we find is the discontinuity in this map. This is just the discontinuity mentioned above in changing from a periodic potential to a non-periodic one.

2. Models defined on the real line

Starting from a superspace action with a superpotential W, we build quantum mechanical models with supersymmetry in the standard way. (See in addition to Witten (1981, 1982) and Salomonson and van Holten (1982) also Lancaster (1984).) They are specified by

$$[x,p]=\mathrm{i} \qquad \{\psi,\psi^{\dagger}\}=1$$
$$Q=(p+\mathrm{i}W')\psi$$
$$Q^{\dagger}=(p-\mathrm{i}W')\psi^{\dagger}$$
$$H=\{Q,Q^{\dagger}\}=\tfrac{1}{2}(p^2+W')^2-\tfrac{1}{2}W''\{\psi,\psi^{\dagger}\}.$$

Here the primes refer to derivatives of $W(x)$ with respect to x. We use the standard representation in which $\binom{1}{0}$ and $\binom{0}{1}$ correspond to zero- and one-fermion states so that

$$2H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+W'^2-W''\sigma_3 \tag{1}$$

and

$$Q=\begin{bmatrix} 0 & 0 \\ -\mathrm{i}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}+W'\right) & 0 \end{bmatrix}.$$

Aiming to study the Witten index for such theories when the spectrum of H has a continuous portion, we consider theories involving the Poschl–Teller or Eckart equation (Poschl and Teller 1983, Eckart 1930, Alhassid *et al* 1983). Thus, we might use

$$W' = L \tanh x \qquad L = 1, 2, 3 \ldots$$

but setting $L = 1$ (and remarking later on higher L) simplifies the presentation at little cost. The bosonic and fermionic sector Schrödinger equations for (1) become

$$\left(-\frac{d^2}{dx^2} + 1 - 2\,\mathrm{sech}^2 x\right)\psi_b = 2E\psi_b \tag{2}$$

$$\left(-\frac{d^2}{dx^2} + 1\right)\psi_f = 2E\psi_f. \tag{3}$$

Equation (2) has a spectrum with one bound state $E = 0$ corresponding to the normalised wavefunction $\psi_b = \mathrm{sech}\, x/\sqrt{2}$ and a continuous spectrum $2E = 1 + q^2$ for each real q, $-\infty < q < \infty$, corresponding to wavefunctions

$$\psi_{bq}(x) \propto \exp(iqx)(\tanh x + iq).$$

Equation (3) is a free-particle Schrödinger equation with solutions

$$\psi_{fq}(x) \propto \exp(iqx)$$

defined for each real q and energy given by $2E = 1 + q^2$. It is clear that the continuum boson and fermion wavefunctions are in one-to-one correspondence with respect to supersymmetry for

$$Q^{+}\begin{pmatrix} 0 \\ \psi_f \end{pmatrix} \propto \begin{pmatrix} \psi_b \\ 0 \end{pmatrix} \qquad Q\begin{pmatrix} \psi_b \\ 0 \end{pmatrix} \propto \begin{pmatrix} 0 \\ \psi_f \end{pmatrix}.$$

Accordingly the Witten index is clearly given by

$$\Delta = 1$$

due to the unpaired $E = 0$ bosonic state. We remark that for $L = 2$ (and similarly for higher L) the bosonic and fermionic sector Hamiltonians involve potentials

$$4 - 6\,\mathrm{sech}^2 x$$

$$3 + (1 - 2\,\mathrm{sech}^2 x).$$

The former, the $L = 2$ Poschl–Teller or Eckart potential, gives rise to the spectrum of $2E$

$$0, 3, 4 + q^2$$

whereas the latter involves the $L = 1$ potential, whose spectrum $0, 1 + q^2$ was used above, shifted by 3, to give $2E = 3, 4 + q^2$. The pairing of all the states but the $E = 0$ bosonic states is thus assured and $\Delta = 1$ for $L = 2$, and, indeed, all L.

Some of the above work can be seen from a different and instructive perspective by reference to the standard techniques of the theory of integrable systems. Setting

$$A = -\frac{d}{dx} + W'$$

we can write the Hamiltonian H of equation (6) as

$$2H=\begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}.$$

Factorisation of an operator into the form $A^\dagger A$ is closely connected with Bäcklund transformations, which are transformations which transform into each other solutions of two differential equations. If y_n is an eigenvector of $A^\dagger A$ belonging to the eigenvalue λ_n, then it is simple to show that $Y_n = Ay_n$ is an eigenvector of $AA^\dagger$ belonging to the same eigenvalue λ_n. In fact $A^\dagger$ is said to generate a Bäcklund transformation. Thus our supersymmetry transformation is none other than a Bäcklund transformation. Further, if $AA^\dagger f=0$, then we have

$$A^\dagger A = AA^\dagger - 2\frac{d^2}{dx^2}\ln f. \tag{4}$$

This gives, for the case $f=\exp(-W)=\operatorname{sech}^L x$ at hand,

$$2H=\begin{bmatrix} -\dfrac{d^2}{dx^2}+L(L+1)\tanh^2 x - L & 0 \\ 0 & -\dfrac{d^2}{dx^2}+L(L-1)\tanh^2 x + L \end{bmatrix}.$$

Clearly the bosonic sector has one more bound state than the fermionic sector. Again, if $AA^\dagger f=0$, then it follows that $AA^\dagger f^{-1}=0$. In other words, i.e. in the language of supersymmetric quantum mechanics, corresponding to the bosonic solution $\binom{\exp(-W)}{0}$, there is a fermionic solution $\binom{0}{\exp(W)}$. Whether or not both such enter the discussion of the quantum theory depends upon their normalisability.

A final point is that we note that solutions of the Poschl–Teller or Eckart equation for arbitrary L can be constructed from the $L=0$ solutions by repeated supersymmetry or Bäcklund transformations.

3. Models defined on the circle C_K

Since calculation of the Witten index seems to be problem-free in the case of theories with only a discrete spectrum, we wish to consider a regularisation of the above models. For this purpose we pass from the models defined on the real line $-\infty<x<\infty$ to related theories defined on a circle C_K of circumference $2K$, which are supersymmetric for all K and which reproduce the models of § 2 in the limit $K\to\infty$. The theories in question involve bosonic potentials:

$$(W')^2 - W'' = L(L+1)k^2\,\mathrm{sn}^2(x,k) - a_L \tag{5}$$

where $\mathrm{sn}(x,k)$ is a Jacobian elliptic function, of modulus k and (real) period $2K$, $K=K(k)$, and a_L is a constant depending on L. Such information on Jacobian elliptic functions as is required in the following can readily be found by reference to a standard textbook, e.g. Whittaker and Watson (1978).

Our choice (5) of potential corresponds to a Schrödinger equation of a well known type, namely Lamé's equation (Whittaker and Watson 1978, Eastham 1973). In fact the superpotential W for (5) is constructed from the lowest energy Lamé polynomial. We note that such Schrödinger equations have been studied before (Alhassid *et al*

1983, Braden 1984) but not in the context of supersymmetry. Although the case of general L can be treated, we consider only the case $L=1$ for ease of presentation.

With $L=1$, W is given by

$$W=\ln \mathrm{dn}(x, K)$$

so that the general supersymmetric result (1) yields the bosonic and fermionic sector Hamiltonians:

$$2H_{\mathrm{b}}=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-k^2(1-2\mathrm{sn}^2(x, k)) \tag{6}$$

$$2H_{\mathrm{f}}=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-k^2(1-2\mathrm{sn}^2(x+K, k)). \tag{7}$$

We remark that (6) follows (1) directly, whereas (7) requires either manipulations based on

$$\mathrm{dn}(x+K, k)=(1-k^2)^{1/2}(\mathrm{dn}(x, k))^{-1}$$

or else via (4) with $f=\mathrm{dn}\, x$. The supersymmetric juxtaposition of (6) and (7) is in any case remarkable. We note firstly that as $K\to\infty$ (and hence $k\to 1$),

$$\mathrm{sn}(x, k)\to\tanh x$$

$$\mathrm{sn}(x+K, k)\to 1$$

so (6) and (7) yield (2) and (3). Since the shift in $x\to x+K$ by half a period $2K$ converts (6) into (7), it is clear that their spectra, in each case discrete, and solutions are in one-to-one correspondence. Also it follows that $\Delta=0$ for all K. So how can be have $\Delta=1$ for the limiting case of $K\to\infty$, studied above? To answer this question consider the solutions of the Schrödinger equations (6) and (7) for $E=0$. They are

$$\psi_{\mathrm{b}}\propto \mathrm{dn}(x, k)$$

$$\psi_{\mathrm{f}}\propto(\mathrm{dn}(x, k))^{-1}$$

which is consistent with remarks made at the end of § 2. Each wavefunction is normalisable for all finite K, as is easily verified. Now, as $K\to\infty$,

$$\mathrm{dn}(x, k)\to\mathrm{sech}\, x$$

and ψ_{b} corresponds to the normalisable $E=0$ bosonic solution found above. Further, the fermionic case gives rise to $\cosh x$, which does satisfy (3). Also Q and $Q^{\dagger}$ both do annihilate $\binom{0}{\cosh x}$. However $\cosh x$ is not normalisable, and so has to be discarded. Moreover, this is necessary only for $K\to\infty$, so that Δ changes discontinuously from 0 to 1 in this limit.

4. Conclusions

We have presented here a class of supersymmetric quantum mechanical models which are soluble both on the line and on the circle. It is a standard regularisation procedure to use fields defined on a circle subject to periodic boundary conditions, and then to pass to the infinite radius limit for the circle. The actual potentials used here arise in the quantum theory of solitons in two-dimensional spacetime, and the regularisation procedure correctly reproduces (Braden 1984) the phase shifts and so on that are used in the computation of quantum corrections to masses (Dashen *et al* 1974).

Our supersymmetric example shows that $\Delta = 0$ for all finite radii, with supersymmetry unbroken, yet $\Delta = 1$ in the infinite radius limit with supersymmetry still unbroken. Such a situation can be understood on quite general grounds. As the forms of H used in § 2 indicate, we are interested in calculating the index of the operator A. Now the index of any elliptic operator on an odd-dimensional compact manifold vanishes, i.e. $\Delta = 0$. Furthermore for an odd-dimensional non-compact manifold, an index can be defined (Callais 1978) on the basis of suitable assumptions and the definition applied to the real line gives $\Delta = 1$.

Finally, we note that the possibility of discontinuous change in the Witten index has been mentioned previously (Davis *et al* 1983a) in connection with supersymmetric QCD. In this theory, for finite quark mass m_q, there are N vacua invariant under supersymmetry, where N is the number of colours, but it is argued that these vacua disappear from the theory as $m_q \to 0$.

References

Alhassid V, Gursey F and Iachello F 1983 *Phys. Rev. Lett.* **50** 873
Braden H W 1984 *Preprint. Periodic Functional Determinants* DAMTP, Cambridge, 84/12
Callias C 1978 *Commun. Math. Phys.* **62** 213
Cecotti S and Girardello L 1982 *Phys. Lett.* **100B** 39
—— 1984 *Nucl. Phys.* B **239** 537
Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262
Damgaard P H and Tsokos K 1984 *Lett. Math. Phys.* **8** 535
Dashen R F, Hasslacher B and Neveu A 1974 *Phys. Rev.* D **10** 4130
Davis A C, Dine M and Selberg N 1983a *Phys. Lett.* **125B** 487
Davis A C, Macfarlane A J and van Holten J W 1983b *Nucl. Phys.* B **216** 493
—— 1984a *Nucl. Phys.* B **232** 473
Davis A C, Macfarlane A J, Popat P C and van Holten J W 1984b *J. Phys. A: Math. Gen.* **17** 2945
Eastham M S P 1973 *The Spectral Theory of Periodic Differential Equations* (Edinburgh: Scottish Academic Press)
Eckart C 1930 *Phys. Rev.* **35** 1303
Girardello L, Imbimbo C and Mukhi S 1983 *Phys. Lett.* **132B** 63
Imbimbo C and Mukhi S 1984a *Nucl. Phys.* B **242** 81
—— 1984b *Nucl. Phys.* B **247** 471
Khare A and Maharana J 1984 *Z. Phys.* C **23** 191
Kihlberg A, Salomonson P and Skagerstam B S 1984 *Preprint, NORDITA*
Lancaster D 1984 *Nuovo Cimento* A **79** 28
Niemi A J and Wijewardhana L C R 1984 *Phys. Lett.* **130B** 389
Nicolai H 1980 *Phys. Lett.* **89B** 341
Poschl G and Teller E 1933 *Z. Phys.* **83** 143
Salomonson P and van Holten J W 1982 *Nucl. Phys.* B **196** 309
Whittaker E T and Watson G N 1978 *A Course of Modern Analysis* (Cambridge: CUP) 4th edn
Witten E 1981 *Nucl. Phys.* B **188** 513
—— 1982 *Nucl. Phys.* B **262** 253